

On hereditarily rational functions

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Abstract

In this paper, we give a short proof of a theorem by Kollar on hereditarily rational functions. This is an answer to his appeal to find an elementary proof which does not rely so much on resolution of singularities. Our approach does not make use of desingularization techniques. Instead, we apply a stronger version of the Łojasiewicz inequality. Moreover, this allows us to sharpen Kollar's theorem.

In his recent paper [3], Kollar introduced a class of continuous rational functions on an algebraic, possibly singular, variety X . Those functions, called hereditarily rational, are defined by the condition that their restrictions to each algebraic subvariety Y of X remain rational. He proved that every continuous rational function on a smooth algebraic variety is hereditarily rational (Proposition 8). Continuous rational functions on smooth algebraic varieties were investigated by Kucharz [4]. Also, Fichou–Huisman–Mangolte–Monnier [2] examined regulous functions on singular algebraic varieties, i.e. those functions which extend to continuous rational functions on an ambient, smooth algebraic variety. For the rudiments of real algebraic geometry, we refer the reader to [1].

The significance of the class of hereditarily rational functions is visible especially in view of a theorem by Kollar [3] which indicates that hereditarily rational functions enjoy the good properties of continuous rational functions on smooth algebraic varieties. Below, we state and prove a sharpening of Kollar's theorem to the effect that one can find an extension F of a given hereditarily rational function f with the same indeterminacy locus.

AMS Classification: 14P05.

Key words: hereditarily rational functions, Łojasiewicz inequality.

Theorem on hereditarily rational functions. *Let X be a subvariety of a smooth algebraic variety M and $f : X \rightarrow \mathbb{R}$ a continuous rational function. Then the two conditions are equivalent:*

- i) f is hereditarily rational;
- ii) f extends to a continuous rational function $F : M \rightarrow \mathbb{R}$.

Moreover, if f is hereditarily rational and regular off an algebraic subvariety $Z \subset X$, then we can find its extension F which is regular on $M \setminus Z$.

Remarks. 1) The additional conclusion about the indeterminacy locus Z is an essential sharpening of Kollar's theorem, because he constructs an appropriate extension F from a regular function which descends to F through a finite sequence of blowings-up biregular over $M \setminus X$. Thus one can only deduce that F is regular on $M \setminus X$. It seems that the use of desingularization or transformation to a normal crossing by blowing up along smooth centers leads to non-trivial modifications both of the indeterminacy locus of a given rational function f and the singular locus of the algebraic variety X .

2) The implication $ii) \Rightarrow i)$ follows immediately from the above-mentioned proposition. The converse implication remains valid when M is an arbitrary, possibly singular, algebraic variety, because it can be embedded into a smooth variety as a closed subvariety. In particular, it holds if X is an algebraic subvariety of a Zariski locally closed subset of \mathbb{R}^n .

3) Kollar's proof of the implication $i) \Rightarrow ii)$ depends heavily on desingularization. He writes that it would be desirable to find an elementary proof, one that does not rely so much on resolution of singularities. This article provides a proof which satisfies, we think, the above demands. We do not make use of resolution of singularities at all. Instead, we apply a version of the Łojasiewicz inequality from [1], Theorem 2.6.6, recalled below. It also holds for continuous definable functions in an arbitrary polynomially bounded, o-minimal structure. Such a version was formulated and applied in our paper [5], which is devoted to carrying over the composite function theorem to the quasianalytic settings.

Łojasiewicz Inequality. *Let $f, g : A \rightarrow \mathbb{R}$ be two continuous semi-algebraic functions on a locally closed, semi-algebraic subset A of \mathbb{R}^n such that*

$$\{x \in A : f(x) = 0\} \subset \{x \in A : g(x) = 0\}.$$

Then there exist a positive integer k and a continuous semi-algebraic function $h : A \rightarrow \mathbb{R}$ such that $g^k = fh$.

Now, we can readily prove the theorem. Clearly, f is hereditarily rational iff there exists a filtration

$$\emptyset = X_0 \subset X_1 \dots X_{m-1} \subset X_m = X,$$

where X_i are algebraic subvarieties of X such that X_i is nowhere dense in X_{i+1} and the restriction of f to $X_{i+1} \setminus X_i$ is a regular function for each $i = 0, 1, \dots, m-1$. The proof is by induction with respect to the dimension of the variety X . It is evident if X is of dimension 0. The induction step comes down to the following

Lemma. *Consider two algebraic subvarieties $A \subset X$ of M and a continuous rational function $f : X \rightarrow \mathbb{R}$ regular on $X \setminus A$ and vanishing on A . Then f extends to a continuous rational function $F : M \rightarrow \mathbb{R}$ regular on $M \setminus A$.*

Obviously, f can be presented as a fraction g/h , where g, h are regular functions on X , $h \geq 0$ and

$$\{x \in X : h(x) = 0\} \subset A.$$

One can find, of course, their regular extensions $G, H : M \rightarrow \mathbb{R}$ such that $H \geq 0$ and

$$\{x \in M : H(x) = 0\} = \{x \in A : h(x) = 0\} \subset A.$$

The rational function G/H is no longer continuous in general.

Consider the blowing-up $\sigma : \widetilde{M} \rightarrow M$ with respect to the ideal (G, H) . Then the pull-back

$$F_1 := \frac{G^\sigma}{H^\sigma} : \widetilde{M} \rightarrow \mathbb{P}_1 \quad \text{with} \quad G^\sigma := G \circ \sigma, \quad H^\sigma := H \circ \sigma,$$

is a regular mapping into the projective line. Let Y be the birational transform of X , $B := \sigma^{-1}(A)$ and C be the Euclidean closure of $Y \setminus B$; C is, of course, a closed semialgebraic subset of \widetilde{M} . Clearly, σ is a biregular mapping of $\widetilde{M} \setminus B$ onto $M \setminus A$.

Observe further that, in the vicinity of C , F_1 is a regular function leading into \mathbb{R} . Indeed, for a point $b \in C \setminus B$, the denominator $H^\sigma(b) \neq 0$ whence $F_1(b) \neq \infty$. On the other hand, if $b \in C \cap B$, a sequence of points $b_\nu \in Y \setminus B$,

$\nu \in \mathbb{N}$, tends to b . But then the sequence $a_\nu := \sigma(b_\nu) \in X \setminus A$, $\nu \in \mathbb{N}$, tends to $a := \sigma(b) \in A$. Consequently,

$$\begin{aligned} F_1(b) &= \lim_{\nu \rightarrow \infty} F_1(b_\nu) = \lim_{\nu \rightarrow \infty} \frac{G^\sigma(b_\nu)}{H^\sigma(b_\nu)} = \lim_{\nu \rightarrow \infty} \frac{G(a_\nu)}{H(a_\nu)} = \\ &= \lim_{\nu \rightarrow \infty} \frac{g(a_\nu)}{h(a_\nu)} = \lim_{\nu \rightarrow \infty} f(a_\nu) = f(a) = 0 \neq \infty. \end{aligned}$$

Hence the pole set $F_1^{-1}(\infty)$ of the mapping F_1 is disjoint with C , as asserted.

Now, take regular functions $P, Q : \widetilde{M} \rightarrow \mathbb{R}$ such that $P, Q \geq 0$ and

$$B = \{x \in \widetilde{M} : P(x) = 0\} \quad \text{and} \quad Y = \{x \in \widetilde{M} : Q(x) = 0\}.$$

Since

$$\{x \in \widetilde{M} : H^\sigma(x)\} \subset B \quad \text{and} \quad Y \cap (\widetilde{M} \setminus C) \subset B \cap (\widetilde{M} \setminus C),$$

it follows from the above version of the Lojasiewicz inequality that there exists a positive integer $k \in \mathbb{N}$ such that

$$P^k \leq H^\sigma \cdot \text{constant} \quad \text{and} \quad P^k \leq Q \cdot \text{constant}$$

locally on $\widetilde{M} \setminus C$; it means that the constants in the above inequalities depend on a neighbourhood of a given point from $\widetilde{M} \setminus C$. Put

$$F_2 := \frac{P^{2k}}{P^{2k} + Q} \cdot F_1.$$

Observe that the first factor takes on value 1 on $Y \setminus B$, whence

$$F_2(y) = (f \circ \sigma)(y) \quad \text{for all } y \in Y \setminus B.$$

On $\widetilde{M} \setminus C$, we get

$$F_2 = \frac{P^k}{P^{2k} + Q} \cdot \frac{P^k}{H^\sigma} \cdot G^\sigma,$$

with the first two factors locally bounded on $\widetilde{M} \setminus C$ and regular off B . Therefore, the restriction of F_2 to $\widetilde{M} \setminus C$ extends continuously through B by taking on zero value, because the third factor G^σ vanishes on B . On the other hand,

in the vicinity of C , F_2 is the product of the regular function F_1 vanishing on B and a bounded (by 1) rational function which is regular off B . Consequently, in the vicinity of C , F_2 extends continuously through B by taking on zero value.

Summing up, we see that $F_2 : \widetilde{M} \rightarrow \mathbb{R}$ is a continuous rational function regular off B and vanishing on B . Consequently, F_2 is constant on the fibres of the proper mapping σ . Hence it descends to a continuous function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ which vanishes on A and is regular off A , $F_2 = F^\sigma$. The last assertion holds since σ is biregular over $\mathbb{R}^n \setminus A$. We have already seen that the functions F_2 and $f \circ \sigma$ coincide on $Y \setminus B$. Therefore the functions F and f coincide on $X \setminus A$, and thus F is an extension of f we are looking for. Hence the lemma follows and the proof is complete.

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